

SOME COMMENTS ON BOSONISATION AND BIPRODUCTS

S. Majid¹

Department of Mathematics, Harvard University
Science Center, Cambridge MA 02138, USA²

+

Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW

Abstract We collect here some less well-known results and formulae about the bosonisation construction which turns braided groups into quantum groups. We clarify the relation with biproduct Hopf algebras (the constructions are not the same), the response to twisting of braided groups and the abstract characterisation via automorphisms of the forgetful functor for the category of (co)modules of a braided group.

Keywords: quantum group – braided group – twisting – bosonisation – biproduct – crossed module – braided reconstruction – colour Lie algebra – supersymmetry

1 Introduction

Recently there has been some interest in the theory of braided groups or Hopf algebras in braided categories[1][2] and the bosonisation functor which relates them to quantum groups[3]. Applications in physics include the spectrum generating quantum groups[4], the construction of inhomogeneous quantum groups[5] and the cross product structure of the quantum double[6][7][8]. Applications in pure mathematics include [9] and [10]. Here we collect some modest results about the construction and address some frequently asked questions. As a novel feature, we give all four possible versions (left modules, right modules, left comodules, right comodules) of the various formulae. This should make the paper quite useful as a reference. We also correct a mathematical confusion in the recent J. Algebra paper [10] where it was assumed incorrectly that bosonisation and the related theory of biproducts[11] have the same input data. We provide a natural counterexample to this assertion. The main new result of the paper is a detailed calculation of the automorphism braided group $BGL_q(2) \bowtie \mathbb{A}_q^2$ of the forgetful functor from the

¹Royal Society University Research Fellow and Fellow of Pembroke College, Cambridge

²During the calendar years 1995 + 1996

category of braided \mathbb{A}_q^2 , which demonstrates explicitly a step from the abstract construction of bosonisation [3] (in comodule form). This construction fully solves a question recently posed by B. Pareigis[12] about ‘hidden symmetries’.

We work over a ground field k and use the usual notations S, ϵ for the antipode and counit, and $\Delta h = h_{(1)} \otimes h_{(2)}$ for the coproduct applied to an element $h \in H$ [13] (summations understood). We use the symbols \bowtie , etc., for cross (or smash) products, \bowtie for cross (or smash) coproducts and \bowtie when both are made simultaneously.

This is the final version of a preprint with similar title and the same mathematical content, circulated in April 1995. Only some material about twisting in Section 3 and assosiativity (10) has been added.

2 Module and Comodule Formulae

One of the joys of Hopf algebras is that for every theorem of a certain general type one gets three theorems free. We will develop this as a formal result in Section 5, but for the moment we merely demonstrate the principle at work for braided groups. Thus, in the original work on braided groups[1] we worked with braided groups B living in the category of left-modules of a quasitriangular Hopf algebra H, \mathcal{R} . We used this version of the theory because it is more familiar for physicists. Less well-known perhaps is the dual version which was also introduced by the author[2], in which we work with B in the category of right-comodules of a dual-quasitriangular Hopf algebra H . Even less well-known is a theory for right modules or left comodules.

By quasitriangular bialgebra or Hopf algebra we mean a Hopf algebra H equipped with invertible $\mathcal{R} \in H \otimes H$ obeying the axioms of Drinfeld[14]

$$\begin{aligned} (\Delta \otimes \text{id})\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \\ \Delta^{\text{op}} &= \mathcal{R}(\Delta)^{-1}. \end{aligned} \tag{1}$$

If we consider \mathcal{R} as a map $k \rightarrow H \otimes H$ and write Drinfeld’s axioms as commuting diagrams, and then reverse all arrows, we have the dual concept of a dual-quasitriangular (or coquasitriangular) Hopf algebra[40][15, Thm. 4.1]. One can then write those axioms out explicitly as a skew bialgebra bicharacter $\mathcal{R} : H \otimes H \rightarrow k$ with respect to which H is quasi-commutative (the dual of Drinfeld’s axioms). Explicitly:

$$\begin{aligned} \mathcal{R}(hg \otimes f) &= \mathcal{R}(h \otimes f_{(1)})\mathcal{R}(g \otimes f_{(2)}), \quad \mathcal{R}(h \otimes gf) = \mathcal{R}(h_{(1)} \otimes f)\mathcal{R}(h_{(2)} \otimes g) \\ g_{(1)}h_{(1)}\mathcal{R}(h_{(2)} \otimes g_{(2)}) &= \mathcal{R}(h_{(1)} \otimes g_{(1)})h_{(2)}g_{(2)}. \end{aligned} \tag{2}$$

Drinfeld requires \mathcal{R} invertible, so we also have to dualise this concept as \mathcal{R} invertible in the convolution algebra of maps $H \otimes H \rightarrow k$.

The rest of Drinfeld's theory can also be dualised. If some results are routinely formulated with diagrams (such as cross product constructions) this is just a matter of reversing arrows. Or if they involve complicated algebra it may be easier to prove the dual version directly. For example, among less well-known results about dual-quasitriangular Hopf algebras one finds in an appendix to [2]:

Proposition 2.1 [2, Prop A.5] *Let H be a dual quasitriangular Hopf algebra. Then the square of the antipode is inner in the convolution algebra $H \rightarrow H$ and hence the antipode is bijective.*

It means that the assumption that the antipode of a dual quasitriangular Hopf algebra H is bijective, which is usually assumed by pure mathematicians e.g. [10, Thm 2.15], should be deleted as superfluous.

Likewise, if the category of modules of a quasitriangular Hopf algebra is braided, then so is the category of comodules of a dual-quasitriangular one. Similarly, this time reflecting all diagrams about a vertical axis, we have both left and right versions of the theory. We recall that a braided category means a braiding Ψ between any two objects. We denote left actions by \triangleright and right actions by \triangleleft . We denote coactions by $v \mapsto v^{(\bar{1})} \otimes v^{(\bar{2})}$, which lives in $H \otimes V$ for a left comodule V and $V \otimes H$ for a right comodule. So the braiding in the four cases is:

$$\begin{aligned} \Psi_L(v \otimes w) &= \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v, & \Psi_R(v \otimes w) &= w \triangleleft \mathcal{R}^{(1)} \otimes v \triangleleft \mathcal{R}^{(2)} \\ \Psi^L(v \otimes w) &= \mathcal{R}(w^{(\bar{1})} \otimes v^{(\bar{1})}) w^{(\bar{2})} \otimes v^{(\bar{1})}, & \Psi^R(v \otimes w) &= w^{(\bar{1})} \otimes v^{(\bar{1})} \mathcal{R}(v^{(\bar{2})} \otimes w^{(\bar{2})}). \end{aligned} \tag{3}$$

The four categories are denoted ${}_H\mathcal{M}$, \mathcal{M}_H , ${}^H\mathcal{M}$ and \mathcal{M}^H , for left, right modules and left, right comodules respectively. This is basically in Drinfeld[14] and more explicitly in [16, Sec. 7] in the module version, among other works from about this time.

Braided groups make sense as algebraic structures within any braided category, but these four categories are the most important. Indeed, super and colour Lie algebras and Hopf algebras have all been studied in isolation for many years. One of the main ideas of the theory of braided groups is (as well as to generalise them to the braided case) to cast these concepts as constructions in one of these four categories. Thus, in [17][18] we introduced the quantum group \mathbb{Z}'_2 which generates the category of supervector spaces as $\text{SuperVec} = {}_{\mathbb{Z}'_2}\mathcal{M}$. This was generalised to the category of anyonic (or \mathbb{Z}_n -graded) vector spaces in 1991[19] and further on to categories generated by Abelian groups equipped with bicharacters[20], which is the setting for colour Lie algebras, except that we do not assume that the bicharacter is skew (in the skew case the

category is symmetric rather than braided and the theory is more straightforward). While one can work with these categories directly, appreciating that they are generated by a quantum group allows one to apply some general Hopf algebra machinery, such as the bosonisation theorem[3]. A common misconception is that this use of quantum groups as generating categories in which we can do such algebra was developed previously in the symmetric, e.g. super case, before the advent of braided groups. As far as I know, it is due to the author and useful, e.g.[4], even in the super case.

Next we turn to the concept of braided group itself, as well as other algebraic constructions in braided categories. A braided group B is like a Hopf algebra except that the coproduct $\underline{\Delta} : B \rightarrow B \otimes B$ is a homomorphism to the *braided tensor product* algebra. The product here involves the braiding Ψ . In concrete cases it is

$$(b \otimes c)(a \otimes d) = b\Psi(c \otimes a)d. \quad (4)$$

In our four preferred categories, the homomorphism property becomes:

$$\begin{aligned} \underline{\Delta}(bc) &= b_{(1)}(\mathcal{R}^{(2)} \triangleright c_{(1)}) \otimes (\mathcal{R}^{(1)} \triangleright b_{(2)})c_{(2)}, & \underline{\Delta}(bc) &= b_{(1)}(c_{(1)} \triangleleft \mathcal{R}^{(1)}) \otimes (b_{(2)} \triangleleft \mathcal{R}^{(2)})c_{(2)} \\ \underline{\Delta}(bc) &= \mathcal{R}(c_{(1)}^{(\bar{1})} \otimes b_{(2)}^{(\bar{1})})b_{(1)}c_{(1)}^{(\bar{2})} \otimes b_{(2)}^{(\bar{2})}c_{(2)}, & \underline{\Delta}(bc) &= b_{(1)}c_{(1)}^{(\bar{1})} \otimes b_{(2)}^{(\bar{1})}c_{(2)}\mathcal{R}(b_{(2)}^{(\bar{2})} \otimes c_{(1)}^{(\bar{2})}), \end{aligned} \quad (5)$$

where $\underline{\Delta}b = b_{(1)} \otimes b_{(2)}$ denotes the braided coproduct. Similarly, the antipode for braided groups is a braided-antihomomorphism:

Proposition 2.2 [17, Fig. 2] *The antipode S of a braided group B is a braided-antihomomorphism in the sense*

$$\underline{S} \circ \cdot = \cdot \circ \Psi \circ (\underline{S} \otimes \underline{S}), \quad \underline{\Delta} \circ \underline{S} = (\underline{S} \otimes \underline{S}) \circ \Psi \circ \underline{\Delta}.$$

Less well-known, however, is that these properties are not the axioms of a braided group but rather they require proof (one of the first non-trivial lemmas in braided group theory). In fact, the antipode axioms are $(\underline{S}b_{(1)})b_{(2)} = \epsilon(b) = b_{(1)}\underline{S}b_{(2)}$ as usual (said diagrammatically in a general category). In our four preferred categories the braided antimultiplicativity property in Proposition 2.2 becomes:

$$\begin{aligned} \underline{S}(bc) &= (\mathcal{R}^{(2)} \triangleright \underline{S}c)(\mathcal{R}^{(1)} \triangleright \underline{S}b), & \underline{S}(bc) &= (\underline{S}c \triangleleft \mathcal{R}^{(1)})(\underline{S}b \triangleleft \mathcal{R}^{(2)}) \\ \underline{S}(bc) &= \mathcal{R}(c^{(\bar{1})} \otimes b^{(\bar{1})})(\underline{S}c^{(\bar{2})})(\underline{S}b^{(\bar{2})}), & \underline{S}(bc) &= (\underline{S}c^{(\bar{1})})(\underline{S}b^{(\bar{1})})\mathcal{R}(b^{(\bar{2})} \otimes c^{(\bar{2})}). \end{aligned} \quad (6)$$

For example, it is perhaps not clear in [10, eqn. (1.9)] that the homomorphism property for $\underline{\Delta}$ is part of the definition, while the property for \underline{S} is not part of the definition but follows as explained above.

The proof of Proposition 2.2 in [17] is by diagrammatic means. As far as I know, it is the only known proof, direct algebraic proofs being impractical. In this diagrammatic method one writes products as $\cdot = \vee$, coproducts as $\Delta = \wedge$ and the braiding as $\Psi = \times$ with inverse $\Psi^{-1} = \oslash$. In this way algebraic information ‘flows’ along braid and knot diagrams[1][17][3], not unlike the manner in which information flows along the wires in a computer. Such wiring diagrams are a standard feature in mathematics and engineering, and have even been used for ordinary Hopf algebras[21]. In all these previous contexts there is no non-trivial operator attached to the crossing of wires. One just wires outputs into inputs without caring about whether one passes over or under another. The novel feature of braided groups is that now, for the first time, crossings represent non-trivial operators Ψ or Ψ^{-1} . The notation makes sense by combining standard ideas about wiring diagrams with the coherence theorem for braided categories[22]. These Ψ correspond to the quasitriangular structure (3), which is precisely the key complication in braided group formulae such as (5) and (6). This is why it was indispensable in [1][17][3].

In these terms we would like to address a further misconception that the braided group theory follows automatically as a generalisation of the older theory of Hopf algebras in symmetric categories. In fact, the symmetric theory follows in a canonical way from the theory of ordinary Hopf algebras since one merely inserts a ‘symmetry’ Ψ in place of transposition in every usual algebraic construction, e.g. [23][24]. The braided case is much more problematic because not only must one choose from Ψ, Ψ^{-1} (they do not coincide in the braided case), but there *may be no consistent choice at all*, i.e. a standard construction for quantum groups may simply get ‘tangled up’ in the braided setting. Even such basic things as the tensor product of braided groups (within the category) and the Jacobi identity in its usual form become tangled up in the braided case; they fail and a new theory is needed[25]. The diagrammatic notation is one of the main tools introduced in the braided theory to help control this problem. There is, however, no automatic way to go from usual results about quantum groups or Hopf algebras in symmetric categories to strictly braided ones. For this reason we really should distinguish carefully between algebraic constructions in symmetric categories, for which there is a canonical procedure to extend general categorical constructions to this case, and the braided case which requires genuinely new work.

Finally, we would like to clarify some notational confusion for which the author is certainly to blame. In the first ‘transmutation’ construction $B(,)$ for braided groups we emphasised the diagonal case $B(H, H)$ which are braided-cocommutative[1] or braided-commutative[18][26] in a certain sense, i.e. like classical groups. These remain some of the most interesting for

conformal field theory [27][28][29], but they are only one case; we have subsequently used the term ‘braided groups’ for any Hopf algebra in a braided category (not only the strict usage as braided (co)commutative ones). For example, the general $B(,)$ transmutation yield quantum-braided groups with braided (dual)quasitriangular structure[2][17]. Thus, [18] emphasised for physicists the braided matrices $B(R) = B(A(R), A(R))$, while more recently the general cases $B(R, Z) = B(A(R), A(Z))$ have proven interesting as well[30][31][32]. Strictly speaking $B(R)$ and $B(R, Z)$ are obtained by transmutation[2] only for suitably nice cases where $A(Z)$ can be replaced by an actual Hopf algebra; but once the formulae are obtained under this assumption, they can all be checked directly along the lines explained in [18] in the diagonal case and [30][31] in the general case. There are now also linear braided groups[5] which are not of this transmutation type at all.

In summary, the braided group theory is different in a fundamental way from the theory of Hopf algebras in symmetric categories (where $\Psi = \Psi^{-1}$). The first examples were introduced (by the author) in both module and comodule form. The module version of the theory[27][1] was presented to the mathematical physics community in Kyoto in May 1990 and in St Petersburg in September 1990. The comodule version[26][2][33] was presented to the Hopf algebra community at the AMS meeting in San Francisco in January, 1991. More general examples of braided groups (not necessarily associated to quantum groups at all) are provided by a more general automorphism construction due to the author[27][2] and independently in the diagonal case to Lyubashenko[28].

3 Bosonisation and Twisting

In this section we recall two important theorems about braided groups, namely the bosonisation construction itself[3, Thm 4.1] and twisting[34][35], giving them now in all of our four categories.

Bosonisation generalises the Jordan-Wigner bosonisation transform for \mathbb{Z}_2 -graded systems in physics, and associates to every Hopf algebra B in the braided category of representations of H an equivalent ordinary Hopf algebra $B \bowtie H$ (left handed cases) or $H \bowtie B$ (right handed cases). If $B \in {}_H\mathcal{M}$ then $B \bowtie H$ is defined as the cross product by the canonical action \triangleright of H (by which B is an object) and a coproduct built from the quasitriangular structure \mathcal{R} and this action. Dually, if $B \in {}^H\mathcal{M}$ then $B \bowtie H$ is defined as the cross coproduct by the canonical coaction of H (by which B is an object) and a product built from the dual quasitriangular structure \mathcal{R} and

the coaction. The explicit formulae in our four categories are:

$$\begin{aligned}
{}_H\mathcal{M} : \quad hb &= (h_{(1)} \triangleright b) h_{(2)}, \quad \Delta b = \underline{b}_{(1)} \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright \underline{b}_{(2)}, \quad Sb = (u \mathcal{R}^{(1)} \triangleright \underline{S}b) S\mathcal{R}^{(2)} \\
\mathcal{M}_H : \quad bh &= h_{(1)} (b \triangleleft h_{(2)}), \quad \Delta b = \underline{b}_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \underline{b}_{(2)}, \quad Sb = (S\mathcal{R}^{(2)}) \underline{S}b \triangleleft \mathcal{R}^{(1)} v \\
{}^H\mathcal{M} : \quad \Delta b &= \underline{b}_{(1)} \underline{b}_{(2)}^{(\bar{1})} \otimes \underline{b}_{(2)}^{(\bar{2})}, \quad bh = \mathcal{R}(h_{(1)} \otimes b^{(\bar{1})}) b^{(\bar{2})} h_{(2)}, \quad Sb = b^{(\bar{1})} (\underline{S}b^{(\bar{2})}) \\
\mathcal{M}^H : \quad \Delta b &= \underline{b}_{(1)}^{(\bar{1})} \otimes \underline{b}_{(1)}^{(\bar{2})} \underline{b}_{(2)}, \quad bh = h_{(1)} b^{(\bar{1})} \mathcal{R}(b^{(\bar{2})} \otimes h_{(2)}), \quad Sb = (\underline{S}b^{(\bar{1})}) b^{(\bar{2})}
\end{aligned} \tag{7}$$

In all cases, H is a sub-Hopf algebra and B a subalgebra. For the antipode, $u = (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ and $v = \mathcal{R}^{(1)}(S\mathcal{R}^{(2)})$.

There are several ways of thinking about this construction. The abstract characterisation [3] of the resulting Hopf algebras is that their (co)modules are monoidally equivalent to the braided B -(co)modules in the braided category. The module case is given explicitly in [3]. The dual theorem, which we will need in Section 6, is:

Proposition 3.1 *cf[3, Thm 4.2] Let H be a dual quasitriangular Hopf algebra and B a Hopf algebra in \mathcal{M}^H . The B -comodules in \mathcal{M}^H can be identified canonically with $H \bowtie B$ -comodules as monoidal categories over ${}_k\mathcal{M}$.*

Proof Cf[3] a B -comodule in the category means a vector space which is an H -comodule and a B -comodule which intertwines the H -coaction. The corresponding coaction of $H \bowtie B$ consists of the B -coaction followed by the H -coaction. Using the properties of a dual quasitriangular structure one finds that this is an identification of monoidal categories (i.e. that the tensor product of comodules is respected). \square

In more concrete terms, there are at least two concrete points of view on the same formulae. The natural one in the context of (7) is that the coproduct in the left module case is the braided tensor product coalgebra $B \underline{\otimes} H_L$ where H_L denotes the left regular representation. Similarly, in the right comodule case the product is the braided tensor product algebra $H^R \underline{\otimes} B$ as in (4), where H^R denotes the right regular corepresentation (H as a right comodule by its coproduct). Similarly for right modules and left comodules.

A second point of view on the same formulae is that the coproduct in the left module case has a cross coproduct form by coaction $\beta(b) = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright b$, which is the induced coaction introduced earlier in [6] – so the bosonisation is manifestly both a cross product and cross coproduct[7]. Similarly, in the right comodule bosonisation cases, the product is a cross product by the induced action $b \triangleleft h = b^{(\bar{1})} \mathcal{R}(b^{(\bar{2})} \otimes h)$ via the dual-quasitriangular structure, etc. This point of view connects with a more general biproduct construction (see Section 4). It is *not*

however, the point of view which captures the key properties of bosonisations. Nor is it the point of view by which the construction was first introduced.

A second important theorem for braided groups is the theory of ‘gauge equivalence’ or twisting of braided groups. Such twisting for Hopf algebras was introduced in the work of Drinfeld[14] and used in [36]; it was extended to braided groups in [34][35]. Its importance for physics is that many systems can appear algebraically different but should really be equivalent in a physical sense. For example, there are currently two ‘twistor’ formulations of q -spacetime based on 2×2 matrices, described by the algebras $R_{21}\mathbf{x}_1\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}_1R$ and $R_{21}\mathbf{u}_1R\mathbf{u}_2 = \mathbf{u}_2R_{21}\mathbf{u}_1R$, respectively [34][18]. The use of such a form as spacetime is due to the author (and works for a general R-matrix), but the choice of su_2 R-matrix recovers previous algebras proposed for ‘Euclidean’ and ‘Minkowski’ q -coordinates (and now as braided groups). The use of this R-matrix form for Minkowski space explicitly occurred in 1992 in [8], with its braided coaddition structure (also in R-matrix form) appearing in 1993 in Meyer’s paper [37]. The point of our discussion is that the twisting theory of braided groups exactly relates the two systems; they are gauge equivalent at the algebraic level, differing up to equivalence only in their choice of $*$ -structure. This makes possible the concept of ‘quantum Wick rotation’ between the two systems[34]. There are many other applications of twisting besides this one.

In the module setting, the data for twisting is $\chi \in H \otimes H$ which is a 2-cocycle for H in the sense $\chi_{23}(\text{id} \otimes \Delta)\chi = \chi_{12}(\Delta \otimes \text{id})\chi$ and $(\epsilon \otimes \text{id})\chi = 1$ and ensures that H_χ with co-product $\Delta_\chi = \chi(\Delta) \chi^{-1}$ and quasitriangular structure $\mathcal{R}_\chi = \chi_{21}\mathcal{R}\chi^{-1}$ is also a quantum group, the twisting of H . In the comodule setting the dual data is $\chi : H \otimes H \rightarrow k$ obeying $\chi(h_{(1)} \otimes f_{(1)})\chi(g \otimes h_{(2)}f_{(2)}) = \chi(g_{(1)} \otimes h_{(1)})\chi(g_{(2)}h_{(2)} \otimes f)$ and $\chi(1 \otimes h) = \epsilon(h)$ and ensures that H_χ with product $h \cdot_\chi g = \chi(h_{(1)} \otimes g_{(1)})h_{(2)}g_{(2)}\chi^{-1}(h_{(3)} \otimes g_{(3)})$ and dual quasitriangular structure $\mathcal{R}_\chi(h \otimes g) = \chi(g_{(1)} \otimes h_{(1)})\mathcal{R}(h_{(2)} \otimes g_{(2)})\chi^{-1}(h_{(3)} \otimes g_{(3)})$ is also a quantum group, the dual-twisting of H . More details are in [38]. Now, if B is a braided group in one of our preferred quantum-group generated braided categories then its twisting B_χ lives in the category generated by the twisted quantum group. In the twisting of braided groups, both the product and coproduct are modified. The formulae in the four cases are

$$\begin{aligned}
H_\chi \mathcal{M} : \quad & b \cdot_\chi c = \cdot (\chi^{-1} \triangleright (b \otimes c)), \quad \underline{\Delta}_\chi = \chi \triangleright \underline{\Delta} \\
\mathcal{M}_{H_\chi} : \quad & b \cdot_\chi c = \cdot ((b \otimes c) \triangleleft \chi^{-1}), \quad \underline{\Delta}_\chi = (\underline{\Delta}) \triangleleft \chi \\
H_\chi \mathcal{M} : \quad & b \cdot_\chi c = \chi^{-1}(b^{(\bar{1})} \otimes c^{(\bar{1})})b^{(\bar{2})}c^{(\bar{2})}, \quad \underline{\Delta}_\chi b = \chi(b_{(1)}^{(\bar{1})} \otimes b_{(2)}^{(\bar{1})})b_{(1)}^{(\bar{2})} \otimes b_{(2)}^{(\bar{2})} \\
\mathcal{M}^{H_\chi} : \quad & b \cdot_\chi c = b^{(\bar{1})}c^{(\bar{1})}\chi^{-1}(b^{(\bar{2})} \otimes c^{(\bar{2})}), \quad \underline{\Delta}_\chi b = b_{(1)}^{(\bar{1})} \otimes b_{(2)}^{(\bar{1})}\chi(b_{(1)}^{(\bar{2})} \otimes b_{(2)}^{(\bar{2})}).
\end{aligned} \tag{8}$$

The braided antipode, unit and counit do not change. The twisting formulae for braided groups

appeared in [34], with the detailed proof that the result is again a braided group appearing in [35].

The twisting of braided groups commutes with their bosonisation. Thus $B_\chi \bowtie H_\chi \cong (B \bowtie H)_\chi$ and $H_\chi \ltimes B_\chi \cong (H \ltimes B)_\chi$ are also (long) theorems proven in [35]. Here χ is viewed in the bosonised algebra in the trivial way in order to make the twisting after bosonisation.

Here we want to mention a possible application of these ideas to colour Hopf algebras and Lie algebras[39]. These can be understood as algebraic structures in the comodule category generated by kG, β where $\beta : G \times G \rightarrow k$ is a bicharacter extended as a dual-quasitriangular structure. Here G is Abelian, and the case usually studied is when $\beta(s, t) = \beta(t, s)^{-1}$ (the skew case). In this case the resulting category (of G graded spaces with transposition defined by β) is symmetric and life is much easier. The point is that some colour Lie algebras, while appearing genuinely different, may be twisting equivalent to usual ones. If so then their algebraic properties will tend to be equivalent as well; one can prove results about them by twisting to the ‘gauge’ where they become ordinary Lie algebras, using a theorem about them there, and twisting back to obtain the corresponding theorem for the original colour Lie algebra. We have:

Proposition 3.2 *If a skew bicharacter $\beta = \chi^2$ for some other skew bicharacter χ , then a colour Lie algebra in the category generated by kG, β is twisting equivalent to a usual Lie algebra.*

Proof We twist kG, β by χ . Then $\beta_\chi(s, t) = \chi(t, s)\beta(s, t)\chi^{-1}(s, t) = \chi(t, s)\chi(s, t) = 1$. The category generated by kG, β therefore twists to the category of G -graded vector spaces with its usual trivial transposition. All constructions in the category twist. We consider the enveloping colour Hopf algebra[23]. It has coproduct $\underline{\Delta}\xi = \xi \otimes 1 + 1 \otimes \xi$ and relations $\xi\eta - \beta(|\xi|, |\eta|)\eta\xi = [\xi, \eta]$ for homogeneous elements of degree $| \cdot |$ of the colour Lie algebra (working here with right kG comodules, say). Twisting gives us the relations $\xi \cdot \eta - \eta \cdot \xi = \chi^{-1}(|\xi|, |\eta|)\xi\eta - \chi(|\xi|, |\eta|)\eta\xi = \chi^{-1}(|\xi|, |\eta|)(\xi\eta - \beta(|\xi|, |\eta|)\eta\xi) = \chi^{-1}(|\xi|, |\eta|)[\xi, \eta] \equiv [\xi, \eta]_\chi$. It is easy to check that if $[\cdot, \cdot]$ obeys the colour Jacobi identity etc. (defined in the obvious way with transposition β) then $[\cdot, \cdot]_\chi$ obeys the usual one. \square

More generally, if $\beta = \beta_0\chi^2$ for some other skew bicharacters β_0, χ then the same calculation shows that a β -colour Lie algebra is twisting equivalent to a β_0 -colour Lie algebra by the same formulae. The simplest among skew bicharacters are those which have values ± 1 , which makes them both skew and symmetric. They are super-like in the sense that the transposition is generalised by ± 1 factors. Many colour Lie algebras can be ‘reduced’ up to twisting to ones of this super-like type.

Example 3.3 Let $G = (\mathbb{Z}/m\mathbb{Z})^n$ and $\beta(s, t) = q^{(s, t)}$ for some antisymmetric $\mathbb{Z}/m\mathbb{Z}$ -valued bilinear form on G and $q^m = 1$, a primitive m -th root of 1. A β -colour Lie algebra is twisting equivalent to an ordinary Lie algebra if $m \equiv 1, 3 \pmod{4}$. It is twisting equivalent to a super-like Lie algebra if $m \equiv 2 \pmod{4}$.

Proof If m is odd then 2 is invertible in $\mathbb{Z}/m\mathbb{Z}$. Hence we can write $\chi(s, t) = q^{\frac{1}{2}(s, t)}$ and have an example of the preceding proposition. If $m \equiv 2 \pmod{4}$, we can write every element of $\mathbb{Z}/m\mathbb{Z}$ uniquely in the form $i \equiv 2j$ or $i \equiv 2j + \frac{1}{2}m$, where $j \in \{0, 1, \dots, \frac{1}{2}m - 1\}$. We do this for the values of the bilinear form on the standard basis elements in the upper-triangular range (i.e. we write our bilinear form as an antisymmetric matrix and consider its upper-triangular entries). This defines two antisymmetric matrices and hence two antisymmetric bilinear forms $(\ , \)_1$, $(\ , \)_0$ such that $(s, t) = 2(s, t)_1 + (s, t)_0$, where $(s, t)_0$ has values in $\{0, \frac{1}{2}m\}$. Then $\beta = \chi^2 \beta_0$ where $\beta_0 = q^{(\ , \)_0}$ and $\chi = q^{(\ , \)_1}$. Here β_0 has values in ± 1 . \square

It seems likely that some ideas of Scheunert[39] about reducing certain commutation factors to super ones could be formulated as a twisting equivalence along similar lines. This remains, however, for further work.

The extension of colour Hopf algebras to the braided (non-skew case) occurred in [19][20]. In the latter we studied the bosonisation of such braided groups as a novel approach to physical quantisation. For example, the braided line bosonises to the quantum plane[20]. At the Lie algebra level, the braided case is much more complicated but can be developed in the framework of [25].

4 Bosonisation and Biproducts are Not the Same

Some years ago, Radford characterised Hopf algebras which are both a cross product and cross coproduct by an action of a Hopf algebra H (what he called ‘biproducts’) as Hopf algebras equipped with a split projection to H [11]. It was shown by the author in [6][7] that the acted-upon object B is in fact a braided group in the braided category ${}^H_H\mathcal{M}$ of crossed modules (when H has bijective antipode). It was also explained that when H is finite dimensional this category is just the braided category of modules ${}_{D(H)}\mathcal{M}$, which was already known by then. Here $D(H)$ is Drinfeld’s quantum double quasitriangular Hopf algebra. Finally, it was shown that when H is quasitriangular then the bosonisation construction can be viewed as an example of this more general construction by means of a certain functor ${}_H\mathcal{M} \rightarrow {}^H_H\mathcal{M}$. These results are all due to the author[6][7]. In this section we show that this functor is not, however, an isomorphism (so

the constructions are not the same), contrary to recent assumptions in the literature [10]. The category ${}^H_H\mathcal{M}$ itself was introduced in a different context in [21]. It is also an example of a more general construction of the ‘Pontryagin dual’[40] or ‘double’ of any monoidal category.

Consider for the moment H finite-dimensional. Drinfeld’s quantum double[14] $D(H)$ is a Hopf algebra containing (in some conventions) $H, H^{*\text{op}}$. So ${}_{D(H)}\mathcal{M}$ just means left H -modules and left $H^{*\text{op}}$ -modules which are compatible in that they respect the cross relations of the double. But a left $H^{*\text{op}}$ -module is trivially the same thing (by evaluation) as a left H -comodule. This is the category ${}^H_H\mathcal{M}$. Because $D(H)$ is quasitriangular, we know that this category is braided when H has bijective antipode, as appreciated independently in [21]. Similarly, the category $\mathcal{M}_{D(H)}$ can be formulated as \mathcal{M}_H^H , consisting of compatible right H action and coaction. Explicitly, the left and right compatibility conditions and braidings are:

$$\begin{aligned} {}^H_H\mathcal{M} : \quad & h_{(1)}v^{(\bar{1})} \otimes h_{(2)}v^{(\bar{2})} = (h_{(1)}\triangleright v)^{(\bar{1})} h_{(2)} \otimes (h_{(1)}\triangleright v)^{(\bar{2})}, \quad \Psi_L(v \otimes w) = v^{(\bar{1})} \triangleleft w \otimes v^{(\bar{2})} \\ \mathcal{M}_H^H : \quad & v^{(\bar{1})} \triangleleft h_{(1)} \otimes v^{(\bar{2})} h_{(2)} = (v \triangleleft h_{(2)})^{(\bar{1})} \otimes h_{(1)}(v \triangleleft h_{(2)})^{(\bar{2})}, \quad \Psi_R(v \otimes w) = w^{(\bar{1})} \otimes v \triangleleft w^{(\bar{2})}. \end{aligned} \quad (9)$$

It should be clear that since we have dispensed with $H^{*\text{op}}$ itself we do not need to assume that H is finite-dimensional. So, associated to any Hopf algebra with bijective antipode one has these left and right crossed module braided categories. We do not actually need the antipode of H but only the inverse or skew antipode, for Ψ to be invertible. And we only require the latter in order to have a standard braided-categorical setting (it is not needed for the actual constructions below). Morphisms in all these categories are maps which intertwine both the module and comodule structures.

Now consider B a braided group in ${}^H_H\mathcal{M}$. The conditions entailed in this ensure that the cross product and cross coproduct $B \bowtie H$ simultaneously by the action and coaction is an ordinary Hopf algebra[6][7]. Conversely, every Hopf algebra with split projection to H is isomorphic to one of the form $B \bowtie H$ for B a Hopf algebra in ${}^H_H\mathcal{M}$. This is the braided version of Radford’s theorem[7]. Similarly for $B \in \mathcal{M}_H^H$ we have $H \bowtie B$ and a Hopf algebra with split projection to H is also isomorphic to one of this form.

Some authors have wondered whether this braided version of Radford’s theorem adds to what was known in [11], apart from some terminology. Here we would like to explain that the answer is affirmative. In fact, knowing that B is a braided group carries much more information than the properties elucidated in [11]; it tells us that the operator controlling the exotic nature of the algebra-coalgebra B is a braiding Ψ , obeying the Yang-Baxter relations. It tells us that the product and coproduct are well-behaved with respect to this operator (functoriality of Ψ), and other key properties of braided groups which are needed to prove such things as the

antimultiplicativity in Proposition 2.2 etc. None of these properties are implicit or hinted at in [11]. There are some examples of ‘exotic’ algebra-coalgebras B in Radford’s paper but without proving such properties as braiding, they were not shown at that time to be braided groups. Finally, we note that the braided version puts B into a category with other objects in it, allowing us to make categorical constructions involving B and other objects.

For example, if B, C are algebras in ${}^H_H\mathcal{M}$ then the braided tensor product $B \underline{\otimes} C$ from (4) is again an algebra in ${}^H_H\mathcal{M}$. Explicitly, it has product $(b \otimes c)(a \otimes d) = b(c^{(\bar{1})} \triangleright a) \otimes c^{(\bar{2})}d$, as explained by the author in [7, Prop. A.2]. This is a generalisation of the usual concept of cross product because the latter can be viewed as $B \rtimes H = B \underline{\otimes} H_{\text{Ad}}^L$, where H_{Ad}^L is an algebra in ${}^H_H\mathcal{M}$ by the adjoint action and left regular coaction. This point of view has been used in the braided case (where H itself is a braided group) by Bespalov[41]. To see how such generalised cross products easily arise, note that $B \rtimes H \in {}^H_H\mathcal{M}$ by the tensor product action and coaction, because it is a braided tensor product in ${}^H_H\mathcal{M}$. If we make a cross product again by this tensor product action of H , associativity of braided tensor products tells us that

$$(B \rtimes H) \rtimes H = (B \underline{\otimes} H) \underline{\otimes} H = B \underline{\otimes} (H \underline{\otimes} H) = B \underline{\otimes} (H \rtimes H) \quad (10)$$

where $H \rtimes H$ is a cross product by the adjoint action. In other words, usual cross products are not closed under associativity but they are when viewed as more general braided tensor products. The biproduct $B \rtimes H$ itself is an algebra in ${}^H_H\mathcal{M}$ according to this. It is also a coalgebra in ${}^H_H\mathcal{M}$ as the braided tensor coproduct $B \underline{\otimes} H_L^{\text{Ad}}$.

We note that although [6][7] emphasised working in the category ${}_{D(H)}\mathcal{M}$, assuming that H is finite-dimensional, each of these papers also explained at the relevant point how to proceed in the infinite-dimensional case using ${}^H_H\mathcal{M}$. Specifically, it was observed in [6] (below Cor. 2.3) that (9) was one of Radford’s principal conditions in [11]. (The supplementary conditions that B is an H -module coalgebra and an H -comodule algebra were omitted in [6, Cor 2.3] but should also be understood.) And it was observed in [7] (in the proof of Prop A.2) that the second of Radford’s principal conditions is the braided group homomorphism property for the braiding in (9). Moreover, we would like to say that it is not the case that [7, Prop. A.2] only asserts the converse direction; the forward direction that $B \in {}^H_H\mathcal{M}$ gives a Hopf algebra $B \rtimes H$ is an integral part of the proposition and is covered in the proof.

Another corollary of the braided version of Radford’s construction was that it exhibited more clearly the connection with bosonisation. This is provided by functors of braided categories when

H is quasitriangular or dual quasitriangular. For our four preferred categories the functors are

$$\begin{aligned}
{}_H\mathcal{M} &\hookrightarrow {}^H_H\mathcal{M}, & (V, \triangleright) &\mapsto (V, \triangleright, \beta), & \beta(v) &= \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright v \\
\mathcal{M}_H &\hookrightarrow \mathcal{M}_H^H, & (V, \triangleleft) &\mapsto (V, \triangleleft, \beta), & \beta(v) &= v \triangleleft \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \\
{}^H\mathcal{M} &\hookrightarrow {}^H_H\mathcal{M}, & (V, \beta) &\mapsto (V, \beta, \triangleright), & h \triangleright v &= \mathcal{R}(v^{(\bar{1})} \otimes h) v^{(\bar{2})} \\
\mathcal{M}^H &\hookrightarrow \mathcal{M}_H^H, & (V, \beta) &\mapsto (V, \beta, \triangleleft), & v \triangleleft h &= v^{(\bar{1})} \mathcal{R}(v^{(\bar{2})} \otimes h)
\end{aligned} \tag{11}$$

The idea in each case is to start with an action or coaction and *induce* from it a compatible coaction or action. As far as I know, the first functor is due to the author in [6, Prop. 3.1] (including the infinite-dimensional case), with the others as right-module or comodule versions of the same result. Because these are functors of braided categories, a braided group $B \in {}_H\mathcal{M}$, say, can be viewed in ${}^H_H\mathcal{M}$. It is clear that the corresponding bosonisation can be viewed as an example of a biproduct from this second point of view.

Recently, this biproduct point of view on bosonisation was emphasised in [10], although attributing the bosonisation construction entirely to Radford[11]. Indeed, the authors assert throughout the paper [10, p.594, eqn (1.16), below Prop 1.15, Remark 1.16] that ${}^H\mathcal{M} = {}^H_H\mathcal{M}$ when H is dual-quasitriangular, so that the constructions are strictly identified (and due to Radford since his paper [11] occurred some years earlier). We refer to the introduction of [10] where the bosonisation papers are not mentioned at all.

We show now that this identification ${}^H\mathcal{M} = {}^H_H\mathcal{M}$ can never hold unless H is trivial.

Proposition 4.1 *Let H be a quasitriangular Hopf algebra with $H \neq k$. Then the functor ${}_H\mathcal{M} \rightarrow {}^H_H\mathcal{M}$ introduced in [6] is never an isomorphism. Likewise, let H be a dual quasitriangular Hopf algebra with $H \neq k$. Then ${}^H\mathcal{M} \rightarrow \mathcal{M}_H^H$ is never an isomorphism.*

Proof This is clear in the finite-dimensional case from the construction in [6], where this functor was introduced as pull back along a Hopf algebra projection $D(H) \rightarrow H$. Since $D(H)$ as a vector space is $H^* \otimes H$, this can never be isomorphism. An isomorphism of categories would, by Tannaka-Krein reconstruction, require such an isomorphism. This is the conceptual reason. For a formal proof which includes the infinite-dimensional case, consider $H \in {}^H_H\mathcal{M}$ by the left regular coaction Δ and left adjoint action. If in the image of the first functor (with H quasitriangular), then $h_{(1)} \otimes h_{(2)} = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} h S \mathcal{R}^{(1)}_{(2)}$ for all h in H . Applying ϵ to the second factor tells us that $h = \epsilon(h)$ for all h , i.e. $H = k$. This object in ${}^H_H\mathcal{M}$ can be in the image of the second functor, but this is iff the dual quasitriangular structure is trivial and H commutative. On other hand, consider $H \in \mathcal{M}_H^H$ by the left regular action and left adjoint coaction. If in

the image of the second functor (with H dual quasitriangular) then $hg = \mathcal{R}(g_{(1)}Sg_{(3)} \otimes h)g_{(2)}$. Setting $g = 1$ tells us that $h = \epsilon(h)$ again, hence $H = k$. This object can be in the image of the first functor, but this is iff the quasitriangular structure is trivial and H cocommutative. \square

This means in turn that general ‘biproducts’ associated to $B \in {}^H_H\mathcal{M}$ are much more general than the Hopf algebras obtained by bosonisation when $B \in {}_H\mathcal{M}$ for H quasitriangular or $B \in {}^H\mathcal{M}$ for H dual quasitriangular, the bosonisations having many key properties not holding for general biproducts.

The author would like to note for the record that [7] was circulated in January 1992 and its original form is archived on ftp.kurims.kyoto-u.ac.jp as kyoto-net/92-02-07-majid. The original version of [10] appeared somewhat later (date received June 1992) and was shown to the author in July 1992 at the L.M.S. Symposium on Non-Commutative Rings in Durham, England. The simpler version with ${}^H_H\mathcal{M}$ appearing in [10, Prop. 1.15] was explained by the author to Susan Montgomery at this time as the correct formulation dual to the module bosonisation [3]. We refer to [10, Rem. 1.16] where the original June 1992 version is described. This is perhaps not clear from the published [10].

5 Dualisation as Convention

The principle that certain types of constructions for Hopf algebras have dual versions is clear but perhaps not as widely accepted as it should be. One often finds dual versions of known theorems presented in the literature as new. In this short section I would like to elevate this principle to a mathematical theorem. After this, it really should not be necessary to publish certain types of theorems four times. I would like to argue in fact that just as it is generally accepted that using a right-handed version of a left-handed result is merely a matter of convention and does not entail a new theorem, so the reversal of arrows in the dual formulation is likewise not more than a matter of convention.

One of the common objections to this point of view from experts is that the Hopf algebras of interest may not be finite-dimensional and so may not have an appropriate dual Hopf algebra. This argument is based, however, on a misconception: it is not any specific Hopf algebra which is being dualised but the theorem itself; its axioms and proofs:

Theorem 5.1 *Let T be a theorem whose premises, proofs and results are expressed by commuting diagrams in the category Vec of vector spaces. Then (i) T^{op} defined by reversing all arrows in T is also a correct theorem in the category Vec . (ii) \bar{T} defined by reflecting all diagrams in a mirror is another correct theorem in Vec .*

Proof (i) Reverse all arrows. In categorical terms we make the construction in the opposite category where arrows are reversed. If the theorem involves assuming a certain element in H , consider it as a map $k \rightarrow H$. If a theorem involves the transposition map, reverse it as the transposition map again. A theorem involving an algebra becomes one involving a coalgebra. A theorem involving an action becomes one involving a coaction, etc. The axioms of a Hopf algebra are self-dual in this way. Thus a theorem involving a Hopf algebra and an action becomes one involving a Hopf algebra and a coaction, etc. (ii) It is assumed that tensor products in Vec are all written horizontally and the reflection is in a vertical axis. In categorical terms we make the construction in the category Vec equipped with the opposite tensor product. Thus, left actions become right actions, etc. Again, the axioms of a Hopf algebra are symmetric in this sense. \square

An obvious example is the theorem that a left H -module algebra B leads to an associative algebra $B \rtimes H$, the cross product. The dual theorem is that a left H -comodule coalgebra C leads to an coassociative cross coproduct $C \rtimes H$.

There is an obvious generalisation to theorems in other categories, e.g to braided group constructions in braided categories. Since these are routinely done in any case by a certain diagrammatic notation (see Section 2), dualisation or left-right reflection is even more routine. In this diagrammatic notation all morphisms are considered pointing generally downwards. The dual and left-right reversed version of a braided group construction is given in this case by simply turning the diagram proofs up-side-down.

The theorem does not mean that emphasising the dual version of a construction is not useful for some application, but it is the application itself which would be new. Note also that there can still be a problem if we want not the dual theorem but the actual dual algebra or coalgebra etc. to a given one, i.e. dual in the sense of Hopf algebra duality (adjunction of maps in a rigid category). This can take rather more work. For example, [8] contains the proof that if A, H are dually paired quantum groups and B, C dually paired braided groups (the correct definition of the latter is not completely obvious, and is not symmetric) then the comodule and module bosonisations $A \ltimes B$ and $C \rtimes H$ are dually paired Hopf algebra. Another version with $A \ltimes C$ and $B \rtimes H$ dually paired is in [35].

6 Braided Reconstruction

Apart from the four opening paragraphs, this section is the same as in the version of the paper circulated in April 1995. We recall the more abstract definition of the bosonisation theorem[3], giving it explicitly in the comodule form. We observe that this provides without any work a

solution to a problem recently posed in [12]: we will see that the ‘hidden symmetry’ coalgebra remarked in [12, Cor 5.7] is in fact a braided group, with the structure of a certain braided group cross coproduct, and we will compute a detailed example.

Let us recall that the first result in the theory of braided groups is establishing their existence. While one can easily write down axioms for them, the main problem, which was not solved until 1989, was establishing that those axioms can be satisfied non-trivially. In a symmetric category there is no problem since one can take the enveloping algebra of a colour or other generalised Lie algebra, for example [23], but this was not at all possible in the braided case. The problem was solved (by the author) by introducing the *automorphism braided group* $\text{Aut}(\omega)$ of a monoidal functor ω [27][2]. A particular categorical realisation corresponding to the case $\omega = \text{id}$ (more precisely, the identity inclusion in a cocompletion) was also considered independently, in the following year, by V. Lyubashenko. Using the automorphism braided group construction we were able to not only prove existence but to actually compute concrete examples of braided groups [18] by means of a construction which we called *transmutation*. If $G \rightarrow H$ is a Hopf algebra map and H is dual-quasitriangular then G transmutes to a braided group $B(G, H) \in \mathcal{M}^H$, obtained abstractly as the automorphism braided group of the push out functor $\mathcal{M}^G \rightarrow \mathcal{M}^H$.

Note that we do not need here the most general version [2] of $\text{Aut}(\omega)$ in which $\omega : \mathcal{C} \rightarrow \mathcal{V}$ is a monoidal functor between a monoidal category \mathcal{C} and a braided one \mathcal{V} . For existence of this braided group one needs a representing object for the natural transformation $\text{Nat}(\omega, \omega \otimes ())$ which requires a degree of cocompleteness and rigidity. In [2] we began for simplicity with the strongest assumption that \mathcal{V} is rigid and cocomplete over \mathcal{C} . Later on in the same paper [2, p.205] we dropped the rigidity assumption in favour of rigidity of \mathcal{C} . Another option is to require that the image of ω is rigid. Each approach has some advantages. The construction [2] itself is independent of these details and proceeds as soon as certain *representability conditions* are satisfied, in whatever way. It is this aspect of the automorphism construction which is also the most useful in practice: after obtaining the required formulae under the most convenient assumptions for representability, one can verify directly the properties of $B(G, H)$ etc. by usual algebraic methods [17][2, Appendix].

Building on this work, [3] introduced the bosonisation procedure as a kind of ‘adjoint’ to transmutation. We have described it concretely in Section 3, but its abstract characterisation is as follows. Proceeding in comodule form: let B be a braided group in $\mathcal{C} = \mathcal{M}^H$. It is known from [2] that its category \mathcal{C}^B of braided B -comodules in \mathcal{C} also has a tensor product (is a monoidal category). By more usual Tannaka-Krein arguments [42][43] one knows that this category is

equivalent to the the usual comodules over a usual Hopf algebra. This is the characteristic property of bosonisation[3, Thm. 4.2] as explained in Proposition 3.1. In the four cases:

$${}_B({}^H\mathcal{M}) = B \bowtie_H \mathcal{M}, \quad (\mathcal{M}_H)_B = \mathcal{M}_{H \ltimes B}, \quad {}^B({}^H\mathcal{M}) = B \bowtie^H \mathcal{M}, \quad (\mathcal{M}^H)^B = \mathcal{M}^{H \ltimes B}. \quad (12)$$

The identifications are the obvious ones of the underlying vector spaces. In order to obtain this result, we passed through a more conceptual argument using cross products in the braided category and transmutation. Thus, both $B(H, H)$ and B are braided groups and the former (say, in the comodule case) coacts on B by a coaction corresponding under transmutation to the coaction of H on B as an object in \mathcal{M}^H . Hence[3] we can make a braided group cross coproduct $B(H, H) \blacktriangleright \ltimes B$ with the braided tensor product algebra. We then recognise such braided group cross (co)products as the transmutation of some ordinary Hopf algebras $H \ltimes B$. In our four categories:

$$\begin{aligned} B(H, B \bowtie H) &= B \bowtie B(H, H), & B(H, H \ltimes B) &= B(H, H) \ltimes B, \\ B(B \bowtie H, H) &= B \bowtie B(H, H), & B(H \ltimes B, H) &= B(H, H) \blacktriangleright \ltimes B, \end{aligned} \quad (13)$$

where the transmutation functor $B(,)$ in each case is the one appropriate to the category. This is the abstract construction[3] of bosonisation and the reason that it has a cross product form combined with a braided tensor coproduct in the module case, or cross coproduct form combined with a braided tensor product in the comodule case, as we have seen in Section 3.

Now we want to observe that this construction solves automatically the question posed in [12], namely what braided group does one reconstruct as $\text{Aut}(\omega)$ when we are given $B \in \mathcal{M}^H$ and the forgetful functor $\omega : (\mathcal{M}^H)^B \rightarrow \mathcal{M}^H$?

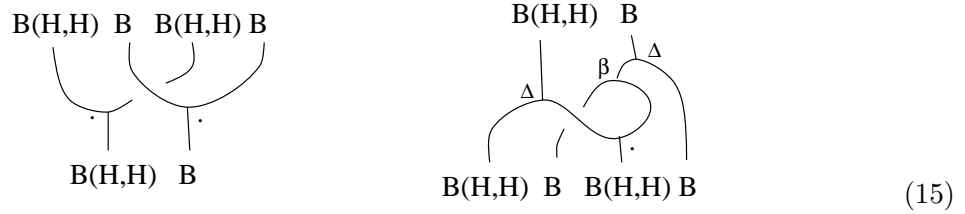
Proposition 6.1 *Let H be dual quasitriangular and B a Hopf algebra in \mathcal{M}^H . Then the forgetful functor ω from B -comodules in \mathcal{M}^H to \mathcal{M}^H has as automorphisms the braided group $B(H, H) \blacktriangleright \ltimes B$ in \mathcal{M}^H . It has the cross coproduct coalgebra and braided tensor product algebra, and is a transmutation of the bosonisation $H \ltimes B$ of B .*

Proof Under the equivalence (12), the forgetful functor ω becomes the functor induced by push-out along the canonical Hopf algebra map $H \ltimes B \rightarrow H$ defined by the counit of B . But the automorphism braided group of a functor induced by push out is exactly the definition of the transmutation construction $B(, H)$. So the answer is exactly the transmutation $B(H \ltimes B, H)$. But the abstract definition of bosonisation in (13) means that this is just $B(H, H) \blacktriangleright \ltimes B$. Indeed, these are exactly the conceptual steps (in comodule form) which led to the bosonisation theory [3] in the first place. \square

This demonstrates how one may use bosonisation theory: we convert our problem for the braided group B to one for its equivalent ordinary Hopf algebra $H \bowtie B$. Explicitly, the braided group $B(H, H)$ associated to H is obtained as the automorphism braided group of the identity functor from \mathcal{M}^H to itself [26][2] and corresponds to $B = k$. Its structure is H as a coalgebra, with the right adjoint coaction and modified product [2]

$$h^{(\bar{1})} \otimes h^{(\bar{2})} = h_{(2)} \otimes (Sh_{(1)})h_{(3)}, \quad h \cdot g = h_{(2)}g_{(2)}\mathcal{R}((Sh_{(1)})h_{(3)} \otimes Sg_{(1)}) \quad (14)$$

in terms of the structure of H . We consider $B(H, H)$ coacting on any B by the same map β by which H coacts on B as an object (the *tautological coaction*). The fact that one can then make a (braided) cross coproduct by this and still obtain a Hopf algebra in the braided category with the braided tensor product algebra structure reflects the fact that $B(H, H)$ is braided-commutative with respect to B in a certain (unobvious) sense introduced in [2]. This was the key idea behind the construction in [3, Sec. 2]. This $B(H, H) \blacktriangleright B$ has product and coproduct defined diagrammatically cf. [3, Sec. 2]



where $\Psi = \times$ is the braiding. The notation is from [1][17][3] and was recalled in Section 2. In our particular case in the category \mathcal{M}^H it means

$$\begin{aligned} (h \otimes b)(g \otimes c) &= h \cdot g^{(\bar{1})} \otimes b^{(\bar{1})} c \mathcal{R}(b^{(\bar{2})} \otimes g^{(\bar{2})}) \\ &= h_{(2)}g_{(3)} \otimes b^{(\bar{1})} c \mathcal{R}((Sh_{(1)})h_{(3)} \otimes Sg_{(2)}) \mathcal{R}(b^{(\bar{2})} \otimes (Sg_{(1)})g_{(4)}) \end{aligned} \quad (16)$$

$$\begin{aligned} \Delta(h \otimes b) &= h_{(1)} \otimes \underline{b_{(1)}}^{(\bar{1})(\bar{1})} \otimes h_{(2)}^{(\bar{1})} \cdot \underline{b_{(1)}}^{(\bar{2})} \otimes \underline{b_{(2)}} \mathcal{R}(h_{(2)}^{(\bar{2})} \otimes \underline{b_{(1)}}^{(\bar{1})(\bar{2})}) \\ &= h_{(1)} \otimes \underline{b_{(1)}}^{(\bar{1})} \otimes h_{(2)} \underline{b_{(1)}}^{(\bar{2})} \otimes \underline{b_{(2)}} \end{aligned} \quad (17)$$

where we evaluated further in terms of B, H . The counit is the tensor product one and there is an antipode as well. The coproduct comes out just the same as for the bosonisation $H \bowtie B$ (the usual cross coproduct by the coaction of H on B as an object in \mathcal{M}^H) because the transmutation procedure $B(, H)$ does not change the coalgebra. This is just the dual of the calculation of $B \bowtie B(H, H)$ in [3, Thm 3.2] for the module version.

These diagrammatic cross products and coproducts where algebraic information ‘flows’ along braids were introduced by the author in [3]. Examples of braided module algebra structures

are the coregular representation which leads to braided-differentiation, and the braided adjoint action [25, Prop. 3.1] which leads to a theory of braided Lie algebras. The comodule versions are the same with diagrams turned up-side-down, e.g. the adjoint coaction [44, Appendix], etc. We refer to [45][46] for many basic results on braided (co)actions used in cross (co)products.

A trivial example of Proposition 6.1 is the case of reconstruction of a super-Hopf algebra $\mathbb{Z}'_2 \blacktriangleleft B$ from the category of super B -modules and its forgetful functor. Here \mathbb{Z}'_2 is the dual of the triangular Hopf algebra introduced by the author in [18, Prop 6.1][17, Ex. 1.1] as generator of the category SuperVec of superspaces with their \mathbb{Z}_2 -graded transposition. This application was generalised in [19] to generate the braided category of anyonic or \mathbb{Z}_n -graded vector spaces introduced there. Unfortunately, in all these examples the adjoint coaction of H is trivial and $B(H, H) = H$ is viewed trivially in \mathcal{M}^H . Hence the algebra structure of $B(H, H) \blacktriangleleft B$ is the usual tensor product one (and its cross coproduct the usual one as well). The result is a braided group in \mathcal{M}^H just because B is. The same applies for Hopf algebras in the braided category of \mathbb{Z} -graded vector spaces.

To give a more non-trivial example, let $q \in k^*$ and $H = GL_q(2)$ defined as $k\langle \alpha, \beta, \gamma, \delta, C^{-1} \rangle$ modulo the relations

$$\begin{aligned} \alpha\beta &= q^{-1}\beta\alpha, & \alpha\gamma &= q^{-1}\gamma\alpha, & \beta\delta &= q^{-1}\delta\beta, & \gamma\delta &= q^{-1}\delta\gamma \\ \beta\gamma &= \gamma\beta, & \alpha\delta - \delta\alpha &= (q^{-1} - q)\beta\gamma, & C &= \alpha\delta - q^{-1}\beta\gamma \end{aligned} \quad (18)$$

essentially as in [14][47] for $SU_q(2)$. We equip it now with dual quasitriangular structure determined by the associated solution R of the quantum Yang-Baxter equations. More precisely (for our application) we take R with a non-standard normalisation as explained in [44], fixed instead by $\mathcal{R}(C \otimes C) = q^6$. Note therefore that one cannot set $C = 1$ as one would for the usual dual quasitriangular Hopf algebra $SU_q(2)$.

The braided group $B(GL_q(2), GL_q(2)) = BGL_q(2)$ is likewise a variant of the braided group $BSU_q(2)$ introduced by the author in [26][18]. We define it as $k\langle a, b, c, d, D^{-1} \rangle$ modulo the relations

$$\begin{aligned} ba &= q^2ab, & ca &= q^{-2}ac, & da &= ad, & bc &= cb + (1 - q^{-2})a(d - a) \\ db &= bd + (1 - q^{-2})ab, & cd &= dc + (1 - q^{-2})ca, & D &= ad - q^2cb \end{aligned} \quad (19)$$

It has ‘matrix’ coproduct $\underline{\Delta}\mathbf{u} = \mathbf{u} \otimes \mathbf{u}$ and $\underline{\Delta}D = D \otimes D$ when we regard the generators as a matrix $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The braided group antipode for \mathbf{u} is as for $BSU_q(2)$ in [26][18] times D^{-1} . The braiding Ψ between the generators is also as listed for $BSU_q(2)$ in [26][18]. This $BGL_q(2)$ lives in $\mathcal{M}^{GL_q(2)}$ with a coaction which has the same ‘matrix conjugation’ form on the generators \mathbf{u} as the right adjoint coaction of $GL_q(2)$.

Let $B = \mathbb{A}_q^2 = k\langle x, y \rangle / (yx - qxy)$ the q -deformed plane with right coaction of $GL_q(2)$ given by transformation of the (x, y) as a row vector by the $GL_q(2)$ generators as a matrix, i.e. $\beta(x) = x \otimes \alpha + y \otimes \gamma$ and $\beta(y) = x \otimes \beta + y \otimes \delta$. One of the first applications of braided groups to physics was to show that this ‘quantum-braided plane’ \mathbb{A}_q^2 is a Hopf algebra in $\mathcal{M}^{GL_q(2)}$ with linear ‘coaddition’ [5]

$$\begin{aligned} \Psi(x \otimes x) &= q^2 x \otimes x, & \Psi(x \otimes y) &= qy \otimes x \\ \Psi(y \otimes y) &= q^2 y \otimes y, & \Psi(y \otimes x) &= qx \otimes y + (q^2 - 1)y \otimes x \\ \Delta x &= x \otimes 1 + 1 \otimes x, & \Delta y &= y \otimes 1 + 1 \otimes y, & \epsilon x &= 0 = \epsilon y, \\ Sx &= -x, & Sy &= -y. \end{aligned} \tag{20}$$

This result is due to the author in [5], where $GL_q(2)$ above is formulated as $\widetilde{SU_q(2)}$, the ‘dilaton’ central extension.

We use the same matrix transformation for the braided coaction of $BGL_q(2)$ on \mathbb{A}_q^2 . Under this, \mathbb{A}_q^2 becomes a right comodule algebra in the braided category [44, Prop. 3.7].

Example 6.2 *The automorphism braided group $BGL_q(2) \blacktriangleleft \mathbb{A}_q^2$ in $\mathcal{M}^{GL_q(2)}$ is generated by $BGL_q(2)$ and the quantum-braided plane \mathbb{A}_q^2 as subalgebras with the cross relations*

$$\begin{aligned} xa &= ax, & ya &= bx(q - q^{-1}) + ay, & xb &= q^{-1}bx, & yb &= qby, & xc &= qcx \\ yc &= (1 - q^{-2})(d - a)x + q^{-1}cy, & xd &= dx, & yd &= dy - q^{-2}(q - q^{-1})bx \end{aligned}$$

It has the matrix coproduct of $BGL_q(2)$ and

$$\underline{\Delta}x = x \otimes a + y \otimes c + 1 \otimes x, \quad \underline{\Delta}y = x \otimes b + y \otimes d + 1 \otimes y$$

extended as a braided group in $\mathcal{M}^{GL_q(2)}$.

Proof The cross relations are exactly the braided tensor product algebra as in (4), computed for the present setting in terms of R in [44, Lem. 3.4]. This gives the relations shown. For the coproduct we know that we have the same form as the cross coproduct by the coaction of $GL_q(2)$ on \mathbb{A}_q^2 but viewed now as a coaction of $BGL_q(2)$. To extend the coproduct to products of the generators we use its braided-multiplicativity, with Ψ determined from the coaction. This was computed in terms of R in [44, Prop. 3.2] and in our case is

$$\begin{aligned} \Psi(a \otimes x) &= x \otimes a + (1 - q^2)y \otimes c, & \Psi(b \otimes x) &= q^{-1}x \otimes b + (q - q^{-1})y \otimes (a - d) \\ \Psi(c \otimes x) &= qx \otimes c, & \Psi(d \otimes x) &= x \otimes d + (1 - q^{-2})y \otimes c \\ \Psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes y\right) &= y \otimes \begin{pmatrix} a & qb \\ q^{-1}c & d \end{pmatrix} \end{aligned} \tag{21}$$

while the braiding $\Psi(x \otimes a) = a \otimes x$ etc., has just the same form as the cross relations already given. It is enough to specify the coproduct and braiding on the generators since the braiding Ψ itself extends ‘multiplicatively’ by functoriality and the Hexagon coherence identities, as explained in [18]. \square

The construction of linear braided groups such as the quantum-braided plane works for general quantum planes associated to suitable matrix data[5]. Another example is the 1-dimensional case $B = \mathbb{A}_q = k[x]$, the braided line[48]. Such ‘linear braided groups’ have been very extensively studied since [5] as the true foundation for q -deformed geometry. See [49] for a review. Their bosonisations were used in [5] to define inhomogeneous quantum groups and are also extensively studied since then. Recently, the bosonisation construction has been generalised so that both input and output are braided groups[41][50].

References

- [1] S. Majid. Braided groups and algebraic quantum field theories. *Lett. Math. Phys.*, 22:167–176, 1991.
- [2] S. Majid. Braided groups. *J. Pure and Applied Algebra*, 86:187–221, 1993. Preprint 1990.
- [3] S. Majid. Cross products by braided groups and bosonization. *J. Algebra*, 163:165–190, 1994. Preprint 1991.
- [4] A. Macfarlane and S. Majid. Spectrum generating quantum group of the harmonic oscillator. *Int. J. Mod. Phys.*, 7(18):4377–4393, 1992.
- [5] S. Majid. Braided momentum in the q -Poincaré group. *J. Math. Phys.*, 34:2045–2058, 1993.
- [6] S. Majid. Doubles of quasitriangular Hopf algebras. *Commun. Algebra*, 19(11):3061–3073, 1991.
- [7] S. Majid. Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group. *Commun. Math. Phys.*, 156:607–638, 1993.
- [8] S. Majid. The quantum double as quantum mechanics. *J. Geom. Phys.*, 13:169–202, 1994. Preprint 1992.
- [9] D. Fischman M. Cohen and S. Westreich. Schur’s double centralizer theorem for triangular Hopf algebras. *Proc. Amer. Math. Soc.*, 122:19–29, 1994.
- [10] D. Fischman and S. Montgomery. A Schur double centralizer theorem for cotriangular Hopf algebras and generalized Lie algebras. *J. Algebra*, 168:594–614, 1994.

- [11] D. Radford. The structure of Hopf algebras with a projection. *J. Algebra*, 92:322–347, 1985.
- [12] B. Pareigis. Reconstruction of hidden symmetries. Preprint December 1994.
- [13] M.E. Sweedler. *Hopf Algebras*. Benjamin, 1969.
- [14] V.G. Drinfeld. Quantum groups. In A. Gleason, editor, *Proceedings of the ICM*, pages 798–820, Rhode Island, 1987. AMS.
- [15] S. Majid. Quantum groups and quantum probability. In *Quantum Probability and Related Topics VI (Proc. Trento, 1989)*, pages 333–358. World Sci. Preprint March 1990.
- [16] S. Majid. Quasitriangular Hopf Algebras and Yang-Baxter Equations. *Int. J. Mod. Phys.*, 5:1–91, 1990.
- [17] S. Majid. Transmutation theory and rank for quantum braided groups. *Math. Proc. Camb. Phil. Soc.*, 113:45–70, 1993. Preprint 1991.
- [18] S. Majid. Examples of braided groups and braided matrices. *J. Math. Phys.*, 32:3246–3253, 1991. Preprint 1990.
- [19] S. Majid. Anyonic quantum groups. In Z. Oziewicz et al, editor, *Spinors, Twistors, Clifford Algebras and Quantum Deformations*, pages 327–336. Kluwer, 1993. Preprint 1991.
- [20] S. Majid. \mathbb{C} -statistical quantum groups and Weyl algebras. *J. Math. Phys.*, 33:3431–3444, 1992.
- [21] D.N. Yetter. Quantum groups and representations of monoidal categories. *Math. Proc. Camb. Phil. Soc.*, 108:261–290, 1990.
- [22] A. Joyal and R. Street. Braided monoidal categories. Mathematics Reports 86008, Macquarie University, 1986.
- [23] D.I. Gurevich. The Yang-Baxter equations and a generalization of formal Lie theory. *Sov. Math. Dokl.*, 33:758–762, 1986.
- [24] B. Pareigis. A non-commutative non-cocommutative Hopf algebra in nature. *J. Algebra*, 70:356, 1981.
- [25] S. Majid. Quantum and braided Lie algebras. *J. Geom. Phys.*, 13:307–356, 1994.
- [26] S. Majid. Rank of quantum groups and braided groups in dual form. In *Proceedings Quantum Groups, St Petersburg, September 1990; Lec. Notes. in Math.*, 1510:79–89, 1992. Springer.

- [27] S. Majid. Reconstruction theorems and rational conformal field theories. *Int. J. Mod. Phys. A*, 6(24):4359–4374, 1991. Preprint 1989.
- [28] V.V. Lyubashenko. Modular transformations for tensor categories. *J. Pure and Applied Algebra*, 98:279–327, 1995.
- [29] V.V. Lyubashenko and S. Majid. Braided groups and quantum Fourier transform. *J. Algebra*, 166:506–528, 1994. Preprint 1991.
- [30] L. Hlavaty. Quantized braided groups. *J. Math. Phys.*, 35:2560–2569, 1994.
- [31] D.M. Lu. Braided matrices $A(R, F)$. *Commun. Alg.*, 22:3009–3013, 1994.
- [32] S. Majid and M.J. Rodriguez-Plaza. Anyonic FRT construction. *Czech. J. Phys.*, 44:1073–1080, 1994.
- [33] S. Majid. Tannaka-Krein theorem for quasiHopf algebras and other results. *Contemp. Maths*, 134:219–232, 1992. Preprint September 1990.
- [34] S. Majid. q -Euclidean space and quantum Wick rotation by twisting. *J. Math. Phys.*, 35:5025–5034, 1994, and hep-th/9401112.
- [35] S. Majid. Quasi- $*$ structure on q -Poincaré algebras. Preprint q-alg/9503014.
- [36] D.I. Gurevich and S. Majid. Braided groups of Hopf algebras obtained by twisting. *Pac. J. Math.*, 162:27–44, 1994. Preprint 1991.
- [37] U. Meyer. q -Lorentz group and braided coaddition on q -Minkowski space. *Commun. Math. Phys.*, 168:249–264, 1995. Preprint July 1993.
- [38] S. Majid. *Foundations of Quantum Group Theory*. Cambridge Univeristy Press, 1995.
- [39] M. Scheunert. Generalized Lie algebras. *J. Math. Phys.*, 20:712–720, 1979.
- [40] S. Majid. Representations, duals and quantum doubles of monoidal categories. *Suppl. Rend. Circ. Mat. Palermo, Ser. II*, 26:197–206, 1991. Preprint 1989.
- [41] Yu. N. Bspalov. Crossed modules and quantum groups in braided categories I,II. Preprints 1994.
- [42] N. Saavedra Rivano. Catégories Tannakiennes. *Springer Lect. Notes in Math*, 265, 1972.
- [43] K.-H. Ulbrich. On Hopf algebras and rigid monoidal categories. *Israel J. Math*, 72:252–256, 1990.

- [44] S. Majid. Quantum and braided linear algebra. *J. Math. Phys.*, 34:1176–1196, 1993.
- [45] S. Majid. Beyond supersymmetry and quantum symmetry (an introduction to braided groups and braided matrices). In M-L. Ge and H.J. de Vega, editors, *Quantum Groups, Integrable Statistical Models and Knot Theory*, pages 231–282. World Sci., 1993, and hep-th/9212151.
- [46] S. Majid. Algebras and Hopf algebras in braided categories. *Lec. Notes in Pure and Appl. Math*, 158:55–105. Marcel Dekker, 1994, and q-alg/9509023.
- [47] L.D. Faddeev, N.Yu. Reshetikhin, and L.A. Takhtajan. Quantization of Lie groups and Lie algebras. *Leningrad Math. J.*, 1:193–225, 1990.
- [48] S. Majid. Solutions of the Yang-Baxter equations from braided-Lie algebras and braided groups. *J. Knot Th. Ram.*, 1995, and hep-th/9312140.
- [49] S. Majid. Introduction to braided geometry and q-Minkowski space. In *Proceedings of the School ‘Enrico Fermi’ CXXVII, Varenna*. IOS Press, Amsterdam, 1995, and hep-th/9410241.
- [50] B. Drabant. Braided bosonization and inhomogeneous quantum groups. Preprint 1994.